Measure of irrationality for certain infinite series

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This is a joint work with Kenji Amano. In 1995, Duverney proved the irrationality of $\sum_{n=0}^{\infty} r^n/(q^n - r)$ for an integer q ($|q| > 1$) and $\mathbb{Q}^{\times} \setminus \{q^i \mid i = 1, 2, \dots\}$ under the some conditions on q and r. But, in applying Duverney's theorem, it is not easy to check that q and r satisfy the conditions in the theorem.

We generalized Duverney's theorem and obtained the measure of irrationality for certain infinite series. In this talk, we will describe the following results.

Theorem 1. *Let* $\mathbf{K}$ *be either* $\mathbb{Q}$ *or an imaginary quadratic field,* q ($|q| > 1$) *be an integer of* $\mathbf{K}$, *and* $r \in \mathbf{K}^{\times}$ ($|r| < |q|$). *Then*

$$\theta = \sum_{n=1}^{\infty} \frac{r^n}{q^n - r^l} \notin \mathbf{K},$$

where $l \geq 1$ *is positive integer. Moreover, for any* $\varepsilon > 0$, *there exists a constant* $C = C(\varepsilon) > 0$ *such that the following inequality*

$$|h_1\theta + h_2| \geq \frac{C}{|h_1|^{5l^2-1+\varepsilon}}$$

holds for all $(h_1, h_2) \in O_{\mathbf{K}}$ ($h_1 \neq 0$).

Example 1. *Let* $q = 3$, $r = 2$, *and* $l = 1, 2, 3$. *Then the numbers*

$$\sum_{n=1}^{\infty} \frac{2^n}{3^n - 2}, \quad \sum_{n=1}^{\infty} \frac{2^n}{3^n - 4}, \quad \sum_{n=1}^{\infty} \frac{2^n}{3^n - 8}$$

are irrational.

Example 2. *Let* ω *be the root of unity satisfying* $\omega^3 = 1$. *Then*

$$\sum_{n=1}^{\infty} \frac{1}{2^n - \omega^n} \notin \mathbb{Q}(\omega).$$